Stability of a neural network model with small-world connections

Chunguang Li*
Institute of Electronic Systems, College of Electronic Engineering, University of Electronic Science and Technology of China, Chengdu, Sichuan, 610054, People's Republic of China

Guanrong Chen†
Department of Electronic Engineering, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong, People's Republic of China

(Received 20 May 2003; revised manuscript received 28 July 2003; published 21 November 2003)

Small-world networks are highly clustered networks with small distances among the nodes. There are many biological neural networks that present this kind of connection. There are no special weightings in the connections of most existing small-world network models. However, this kind of simply connected model cannot characterize biological neural networks, in which there are different weights in synaptic connections. In this paper, we present a neural network model with weighted small-world connections and further investigate the stability of this model.

DOI: 10.1103/PhysRevE.68.052901 PACS number(s): 87.18.Sn, 89.75.Hc, 84.35.+i, 05.45.−a

A great deal of research interest in the theory and applications of small-world networks has arisen [1–8] since the pioneering work of Watts and Strogatz [9]. Some common properties of complex networks, such as Internet servers, power grids, human communities, and disordered porous media, are mainly determined by the type of connection among their vertices or nodes. Among various networks, one extremal case is a regular network with a high degree of local clustering and a large average distance, while the other extremal case is a random network with negligible local clustering and a small average distance. In between the two extremes there are small-world networks, which are a special type of complex network with a high degree of local clustering as well as a small average distance.

Many biological neural networks are small-world networks [10–13]. In most existing literature about small-world networks, there are no weightings in the internal connections of nodes. However, there are many networks, particularly biological neural networks, having weights associated with the connections. These connection-weighted networks cannot be described and characterized by the previously proposed small-world network models. Of particular interest is [13], where small-world neural networks have random weights in their connections; the paper studied the cluster coefficient and the characteristic path of such networks. In this paper, we use dynamical equations to describe a connection-weighted small-world neural network model and then further study its stability with respect to the network topology.

For this purpose, consider a neural network with N neurons described by

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -Au(t) + Wg(u(t)) + I, \tag{1}$$

where $u(t)=[u_1(t),u_2(t),\ldots,u_N(t)]^T$ is the neuron state vector, $A=\mathrm{diag}\{a_1,a_2,\ldots,a_N\}$ is a positive diagonal matrix, $g(u)=[g_1(u_1),g_2(u_2),\ldots,g_N(u_N)]^T$ denotes the neuron activation functions with $g(0)=0$, $I=[I_1,I_2,\ldots,I_N]^T$ is a constant vector, $W=\{w_{ij}\}_{N\times N}$ is the connection-weighting matrix, in which, as in [13], w_{ij} is defined as follows: if there is a connection between neuron i and neuron j $(j\neq i)$, then there is a uniform random distribution $w_{ij}=w_{ji}$ in the connection, with values $0<w_{ij}=w_{ji}<1$; otherwise, $w_{ij}=w_{ji}=0$ $(j\neq i)$. The diagonal elements of W are all zeros, which means that there are no self-connections of nodes within the network. Throughout this paper, we assume that each activation function in Eq. (1) satisfies the following sector condition: There is a real constant, $k\in R$, such that

$$0\leqslant \frac{g_j(x)-g_j(y)}{x-y}\leqslant k, \quad \forall\ x,y\in R, \quad j=1,2,\ldots,N.$$

This type of neural network with full and regular connections has been extensively investigated. However, neural networks with small-world connections have not been thoroughly studied, particularly with respect to their stabilities. For example, it is not clear whether or not the small-world neural networks are easier to stabilize than the fully connected ones. In this paper, we address this question by carefully studying the model (1).

In the following, we always shift the equilibrium u^* of network (1) to the origin. By making the transform $x(t)=u(t)-u^*$, we convert model (1) to the following:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -Ax(t) + Wf(x(t)), \tag{2}$$

where $f_j(x_j(t))=g_j(x_j(t)+u_j^*)-g_j(u_j^*)$, $j=1,2,\ldots,N$. Note that f_j also satisfies a sector condition in the form of

$$f_j(x_j)[f_j(x_j)-kx_j]\leqslant 0, \quad j=1,2,\ldots,N. \tag{3}$$

For this small-world neural network model, we have the following theoretical result.

*Email address: cgli@uestc.edu.cn
†Email address: gchen@ee.cityu.edu.hk

Lemma. Let $\lambda_{\max}(M)$ denote the largest eigenvalue of matrix M. If $\lambda_{\max}(-A/k+W)<0$, then network (2) is asymptotically stable about the origin.

Proof. Select a Lyapunov function as

$$V(x(t))=\sum_{i=1}^{N}\int_0^{x_i} f_i(s)ds.$$

Using the method presented in [14,15], it is easy to verify that $V(x(t))$ is a Lyapunov function. In fact, if we define the following function:

$$G_i(u)=\min\left\{\int_0^{u} f_i(\theta)d\theta, \int_0^{-u} f_i(\theta)d\theta\right\},$$

we have $G_i(0)=0$, $G_i(u)=G_i(|u|)$; for $r\in R^+$, $G_i(r)>0$, $r>0$; $G_i(r)\to+\infty$, $r\to+\infty$. Let $G=\min\{G_i\}$. Then

$$\begin{aligned} V(x)&=\sum_{i=1}^{N}\int_0^{x_i} f_i(s)ds \\ &\geqslant \sum_{i=1}^{N} G_i(x_i) \\ &=\sum_{i=1}^{N} G_i(|x_i|) \\ &\geqslant \sum_{i=1}^{N} G(|x_i|)\geqslant G(|x|). \end{aligned}$$

Therefore, we have achieved a lower bound by a positive, radially unbounded function. It is then easy to verify [15] that

$$G(|x(t)|)\leqslant V(|x(t)|)\leqslant qk\|x\|^2, \quad q>1.$$

The derivative of $V(x)$ along the trajectories of Eq. (2) is, by using Eq. (3),

$$\begin{aligned} \dot{V}(t)&=[f_1(x_1),\ldots,f_N(x_N)][\dot{x}_1,\ldots,\dot{x}_N]^T \\ &=f^T(x(t))[-Ax(t)+Wf(x(t))] \\ &=-f^T(x(t))Ax(t)+f^T(x(t))Wf(x(t)) \\ &\leqslant f^T(x(t))\left[-\frac{A}{k}+W\right]f(x(t)) \\ &\leqslant \lambda_{\max}\left(-\frac{A}{k}+W\right)\|f(x(t))\|^2. \end{aligned}$$

Therefore, if $\lambda_{\max}(-A/k+W)<0$, then we have $\dot{V}(x(t))<0$, implying that network (2) is asymptotically stable about the origin.

Because A is a diagonal positive matrix, it is easy to deduce the following corollary.

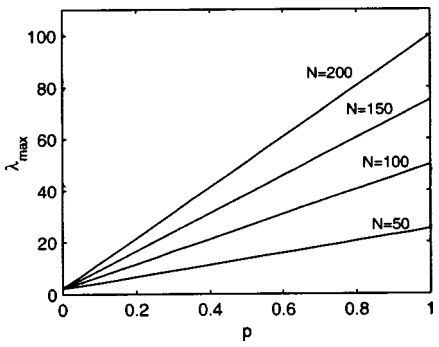

FIG. 1. Maximum eigenvalue of W with respect to p.

Corollary. If $\lambda_{\max}(W)<\min\{a_i/k\}$, then network (2) is asymptotically stable about the origin.

Although we can derive some less conservative stability conditions for Eq. (2), we use only the above results in this paper, because they are very simple and easy to verify. Further, because these conditions use only the maximum eigenvalue of the connection matrix W, we can ''average'' them when using statistical methods to investigate the properties of the connection matrix, as further explained in the following.

Aiming to describe a transition from a regular network to a random network, [9] introduced an interesting model, now referred to as the small-world (SW) network. The original SW model can be described as follows. We take a one-dimensional lattice of N vertices arranged in a ring with connections only between nearest neighbors. We then ''rewire'' each connection with probability p. Rewiring in this context means randomly reconnecting the whole lattice, with the constraint that no two different vertices can have more than one connection between them, and no vertex can have a connection with itself.

Note, however, that it is quite possible for the SW model to be broken into unconnected clusters. This problem can be resolved by a slight modification of the SW model, suggested by Newman and Watts (NW) recently [1]. In the NW model, we do not break any connection between any two nearest neighbors. Instead, we add with probability p a connection between each unconnected pair of vertices. Likewise, we do not allow a vertex to be coupled to another vertex more than once, or a vertex to be coupled with itself. For $p=0$, this reduces to the originally nearest-neighbor coupled network; for $p=1$, it becomes a globally coupled network. Here, we are interested in the NW model starting from a nearest-neighbor lattice with four neighbors and a connection-adding probability $0<p<1$.

From a coupling-matrix point of view, network (2) with small-world connections evolves according to the rule that, in the nearest-neighbor coupling matrix W, if $w_{ij}=0$, we set $w_{ij}=w_{ji}=w$ with probability p and a uniformly randomly distributed weight $0<w<1$. We denote the new small-world coupling matrix by $W(p,N)$ and let $\lambda_{\max}(p,N)$ be its largest eigenvalue. According to the corollary above, if $\lambda_{\max}(p,N)<\min\{a_i/k\}$, then the corresponding small-world neural network is asymptotically stable about its zero state.

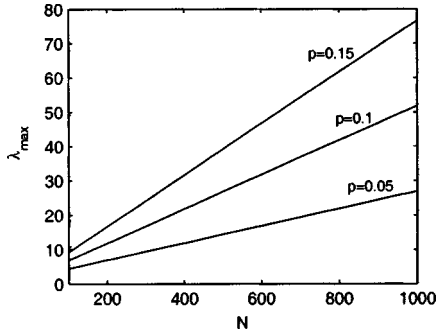

FIG. 2. Maximum eigenvalue of W with respect to N.

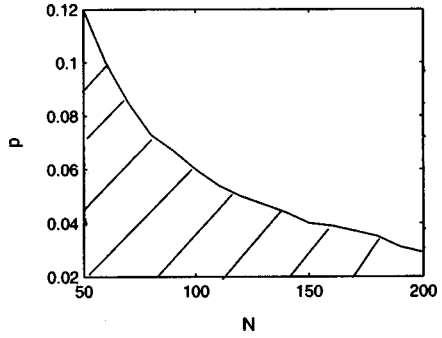

FIG. 3. Stability zone of the network in the given example.

Clearly, the network stability depends on the probability p, so it is more practical to investigate the statistical properties of the connection matrix W. It is easy to see that the mathematical expectation of the number of neurons that are connected to each neuron, i.e., the number of nonzero entries in each row of $W(p,N)$, is $n_c=4+(N-5)p$. Because of the uniform random distribution of the weight values, it is also easy to see that the mathematical expectation of the sum of entries in each row of $W(p,N)$ is $0.5[4+(N-5)p]$, where $N\geqslant 5$ (the smallest neuron number of the nearest-neighbor lattice with four neighbors). Thus, by Lemma 2 of [16], we can calculate the mathematical expectation of $\lambda_{\max}(p,N)$, which is $0.5[4+(N-5)p]$, where $N\geqslant 5$. Hence, the small-world neural network is asymptotically stable about its zero state, in the sense of mathematical expectation, if $0.5[4+(N-5)p]<\min\{a_i/k\}$, $N\geqslant 5$. This also means that the small-world neural network is asymptotically stable in the sense of mathematical expectation if the number of connections of each neuron n_c in the networks is $n_c<2\min\{a_i/k\}$.

Figures 1 and 2 show the numerical values of $\lambda_{\max}(p,N)$ as a function of the probability p and the number of neurons N. In these figures, for each pair of values p and N, $\lambda_{\max}(p,N)$ is obtained by averaging the results of 20 runs. From the above analysis and these figures, we can see that (1) for any given value of $N\geqslant 5$, $\lambda_{\max}(p,N)$ increases almost linearly from about 2 to about $(N-1)/2$ as p increases from 0 to 1; (2) for any given value of $p\in(0,1]$, $\lambda_{\max}(p,N)$ increases almost linearly to $+\infty$ as N increases to $+\infty$. The above results imply that, if the given matrix A satisfies $\min\{a_i/k\}>2$ then (1) for any given $N\geqslant 5$, there exists a critical value p^* such that if $0\leqslant p\leqslant p^*$ then the small-world neural network is asymptotically stable about its zero state (in the sense of mathematical expectation); and (2) for any given $p\in(0,1]$, there exists a critical value N^* such that if $5\leqslant N\leqslant N^*$ then the small-world neural network is asymptotically stable about its zero state (in the sense of mathematical expectation).

Clearly, neural networks with small-world connections are easier to stabilize than their regular fully connected counterparts.

Next, we consider an example of network (1), with the constant vector $I=0$, $A=\mathrm{diag}\{5,5,\ldots,5\}$, and the activation condition $g(\cdot)=\tanh(\cdot)$, which also satisfies condition (3).

From the above results, we know that for any given N (or any given p), there exists a corresponding p (or a corresponding N) that guarantees the stability of the network. The hatched zone in Fig. 3 shows the values of p and N that ensure the stability of this small-world neural network. In this example, the averaged number of connections of each neuron in various configurations is $n_c=9.8592$. This result also coincides with the above analysis.

In summary, a small-world neural network model has been presented and its stability has been analyzed. An analytical expression has been derived that establishes the relationship between the stability and the probability p. Because there are many biological neural networks that present small-world connections, the results obtained in this paper are practical and should be useful for further studies of this kind of neural network model.

We acknowledge support from the National Natural Science Foundation of China under Grant No. 60271019, the Hong Kong Research Grants Council under the CERG Grant No. CityU 1004/02E, and the Youth Science and Technology Foundation of UESTC under Grant No. YF020207.

[1] M. E. J. Newman and D. J. Watts, Phys. Rev. E **60**, 7332 (1999).
[2] C. F. Moukarzel, Phys. Rev. E **60**, 6263 (1999).
[3] M. E. J. Newman, C. Morre, and D. J. Watts, Phys. Rev. Lett. **84**, 3201 (2000).
[4] X. F. Wang and G. Chen, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 187 (2002).
[5] H. Hong, M. Y. Choi, and B. J. Kim, Phys. Rev. E **65**, 026139 (2002).
[6] M. Barahona and L. M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).
[7] P. M. Gade and C. K. Hu, Phys. Rev. E **62**, 6409 (2000).
[8] D. J. Watts, *Small-Worlds: The Dynamics of Networks Between Order and Randomness* (Princeton University Press, Princeton, NJ, 1999).
[9] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[10] J. G. White, E. Southgate, J. N. Thompson, and S. Brenner,

Philos. Trans. **314**, 1 (1986).
[11] T. B. Achacoso and W. S. Yamamoto, *AY's Neuroanatomy of C. Elegans for Computation* (CRC, Boca Raton FL, 1992).
[12] L. F. Lago-Fernández, R. Huerta, F. Corbacho, and J. A. Sigüenza, Phys. Rev. Lett. **84**, 2758 (2000).
[13] C. Aguirre, R. Huerta, F. Corbacho, and P. Pascual, Lect. Notes Comput. Sci. **2415**, 27 (2002).
[14] M. Joy, J. Math. Anal. Appl. **232**, 61 (1999).
[15] X. Liao, G. Chen, and E. N. Sanchez, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **49**, 1033 (2002).
[16] C. W. Wu and L. O. Chua, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **42**, 430 (1995).